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**HYDROMAGNETIC  
WAVE PROPAGATION  
IN A CONSTANT  
DIPOLE  
MAGNETIC FIELD**

**FINAL REPORT**

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IN A CONSTANT DIPOLE MAGNETIC FIELD 142

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SYLVANIA ELECTRONIC SYSTEMS

A Division of Sylvania Electric Products Inc.

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## ABSTRACT

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The study begins with some formal apparatus of the theory (non-linear). The analogue of Clebsch's transformation of the hydrodynamical equations allows a useful representation for the magnetic field; this representation leads in turn to some canonical equations for the motion of magnetic field of great theoretical interest.

A theorem of decomposition of the magnetic field similar to the Cauchy-Stokes decomposition theorem in hydrodynamics is also here presented. The rest of the report deals with hydromagnetic wave propagation in (i) a constant and uniform magnetic field; (ii) in a constant dipole.

The effect of compressibility is especially investigated in the case of a constant and uniform magnetic field.

The disturbance is specified in terms of vorticity and current density. It appears that the compressibility of a medium acts as a wave filter discriminating between components of vorticity (and current density) and passing only those directed along the (undisturbed) magnetic field.

The case of a dipole magnetic field presents a singular importance, in view of its applications to geophysical phenomena, and is discussed in some great detail. Dungey in his remarkable report of 1954 (The Pennsylvania State University, Ionosphere Research Laboratory, Scientific Report No. 69) has already discussed the electrodynamic behavior of the Outer Atmosphere in the presence of a constant-dipole magnetic field. In this study, however, the problem is approached from a different point of view; the magnetohydrodynamic behavior of the fluid is discussed in terms of vorticity and current density. The equations obtained are complicated, however, and solutions are discussed only at large distances from the center of the dipole.

The study of hydromagnetic wave propagation in a dipole is preceded by a chapter where the geometry of lines of force is presented.

Author

## FORWARD

The material in this report summarizes the results of researches undertaken by the writer on the subject of Hydromagnetic Wave Propagation in an electrically conducting fluid of infinite extent embedded in a constant dipole magnetic field. This work has been supported by a contract with the National Aeronautics and Space Administration (Contract No. NASr-18). The little time allowed to us to investigate the subject and the complexity of the problem have permitted only an exploratory effort. The phenomena, as indicated by mathematical analysis, are no doubt very complicated and considerable more effort is required in this direction. It is hoped, however, that this pioneer effort may serve as an introduction to subsequent detailed investigations.

The author takes this occasion to express his gratitude to Dr. Robert Jastrow for his interest in and support of this work.

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## INTRODUCTION

The theory of magnetohydrodynamic waves is the creation of Alfvén and Walén. In his celebrated paper of 1942<sup>(1)</sup> Alfvén has shown that waves can travel along magnetic lines of force in a conducting material; see also Reference 2. Walén,<sup>(24)</sup> in 1944, has discussed in some detail these waves and has given the magnetohydrodynamic equations starting with the principle of conservation of energy.

To these pioneer efforts other savants (Spitzer<sup>(19)</sup>, Cowling<sup>(7,8)</sup>, Grad<sup>(15)</sup>, Lighthill<sup>(17)</sup>, and MacDonald<sup>(18)</sup>) added their researches, resulting in the elegant theory of magnetohydrodynamic waves as presented, for instance, in Lighthill's great memoir of 1961. My earlier work consists mainly in analysis for the case of a compressible fluid.<sup>(3,5)</sup>

All this work presupposes a uniform magnetic field. Non-uniformity of the magnetic field effects the theory both because the wave velocity varies in magnitude and direction and because new forces are introduced. The case of a dipole magnetic field is of particular importance, in view of its applications to geophysical phenomena. This revivifies Störmer's work<sup>(22)</sup> and other aurora theories. It is of great importance indeed to know up to what an extent Störmer's theory may be improved by using the hydromagnetic approach, that is to say, assuming a fluid mechanics continuum approximation.

Hydromagnetic wave propagation in a dipole magnetic field has already been discussed in some detail by Dungey.<sup>(9)</sup> Our approach is, however, more general and sets up the magnetohydrodynamic equations in the Outer Atmosphere at large distances from the earth.

In a subject which is developing so rapidly, this problem cannot be discussed with any approach to finality, but this did not seem to be a reason against writing these pages.

In Chapter I we rapidly review the basic equations of magnetohydrodynamics. The analogue of Clebsch's transformation of the hydrodynamical equations allows a useful representation for the magnetic field; this representation leads in turn to some canonical equations for the motion of magnetic field of great theoretical interest. The Chapter is concluded with a theorem of decomposition for the rate of change of the magnetic field similar to the Cauchy-Stokes decomposition theorem in

hydrodynamics. Chapter II presents the theory of magnetohydrodynamic waves in the presence of a uniform magnetic field. In Chapter III we discuss in some detail the geometry of dipole magnetic lines of force. Chapter IV deals with magnetohydrodynamic waves in a constant dipole magnetic field.

## CHAPTER I

## BASIC CONCEPTS - THE EFFECT OF A MAGNETIC FIELD

1. The Basic Equations of Magnetohydrodynamics

When the displacement currents may be neglected, Maxwell's equations are

$$\text{curl } \underline{H} = 4\pi \underline{J} \quad (1)$$

$$\text{curl } \underline{E} = -\mu_e \frac{\partial \underline{H}}{\partial t} \quad (2)$$

$$\text{div } \underline{H} = 0 \quad (3)$$

where the electromagnetic variables are measured in electromagnetic units,  $\underline{E}$  and  $\underline{H}$  are the intensities of the electric and magnetic fields,  $\underline{J}$  is the current density, and  $\mu_e$  is the magnetic permeability. To complete the equations for the field, we need an equation for the current density.

Consider an electrically conducting fluid which has a conductivity  $\sigma$  and executes motions described by the velocity  $\underline{v}$ . The electric field it will experience is  $\underline{E} + \mu_e \underline{v} \times \underline{H}$ , thus

$$\underline{J} = \sigma(\underline{E} + \mu_e \underline{v} \times \underline{H}) \quad (4)$$

The equations (1) - (4) incorporate the effect of fluid motions on the electromagnetic field. The inverse effect of the field on the motions results from the ponderomotive which the fluid elements experience by virtue of their carrying currents across magnetic lines of force. This is the Lorentz force given by

$$\underline{\mathcal{L}} = \mu_e \underline{J} \times \underline{H} = \frac{\mu_e}{4\pi} \text{curl } \underline{H} \times \underline{H} \quad (5)$$

Including this force among other forces acting on the fluid, we have the equation of motion

$$\rho \frac{d\underline{v}}{dt} = \text{div } \underline{P} + \rho \underline{X} + \mu_e \underline{J} \times \underline{H} \quad (6)$$

where  $\rho$  is the density,  $\underline{P}$  is the total stress tensor and  $\underline{X}$  represents the external forces of non-electromagnetic origin.



In tensor notation, this equation can be written

$$\rho \left[ \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right] - \frac{\mu_e}{4\pi} H_j \frac{\partial H_i}{\partial x_j} = \frac{\partial P_{ij}}{\partial x_j} + \rho X_i - \frac{\partial}{\partial x_i} \left[ \frac{\mu_e H^2}{4\pi} \right] \quad (7)$$

where explicitly

$$P_{ij} = -p\delta_{ij} + 2\mu e_{ij} - \frac{2}{3} \delta_{ij} e_{kk} \quad (8)$$

and where  $p$  is the isotropic pressure,  $\mu$  is the coefficient of viscosity, and  $e_{ij}$  is the rate of deformation given by

$$e_{ij} = \frac{1}{2} \left[ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right] \quad (9)$$

For an incompressible fluid in which  $\mu$  is constant and the forces  $X$  derive from a potential  $-\Omega$ , the equation of motion (7) simplifies to

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} - \frac{\mu_e H_j}{4\pi\rho} \frac{\partial H_i}{\partial x_j} = - \frac{\partial}{\partial x_i} \left[ \Omega + \frac{p}{\rho} + \frac{\mu_e H^2}{4\pi\rho} \right] + \nu \nabla^2 v_i \quad (10)$$

where  $\nu = \mu/\rho$  denotes the kinematic viscosity.

In the general case, the equation of motion (7) has to be supplemented with the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) = 0 \quad (11)$$

and the heat equation. We shall not write down the heat equation assuming in this study that our variables do not depend of temperature.

## 2. The Equation of Motion for the Magnetic Field

We shall now obtain an equation of motion for the magnetic field. In view of further developments, it is convenient to introduce here the vector potential  $\underline{A}$  and the electrostatic potential  $\phi$ , writing in the usual way

$$\underline{H} = \text{curl } \underline{A} \quad (12)$$

$$\text{div } \underline{A} = 0 \quad (13)$$

$$\underline{E} = - \frac{\partial \underline{A}}{\partial t} - \text{grad } \phi \quad (14)$$

We then obtain according to equation (4)

$$\underline{J} = \sigma \left[ - \frac{\partial \underline{A}}{\partial t} - \text{grad } \phi + \mu_e \underline{v} \times \text{curl } \underline{A} \right] \quad (15)$$

Substitution into equation (1) gives

$$\frac{\partial \underline{A}}{\partial t} = \underline{v} \times \text{curl } \underline{A} - \text{grad } \phi - \nu_e \text{curl curl } \underline{A} \quad (16)$$

where  $\nu_e = (4\pi\mu_e\sigma)^{-1}$  will be designated as the magnetic viscosity (see Elsasser,<sup>(13)</sup> page 21). It may be noted that  $\nu_e$  like  $\nu$  is of dimensions  $\text{cm}^2 \text{sec}^{-1}$ .

Taking the curl of terms of equation (16) and assuming  $\nu_e$  a constant, we obtain (in Cartesian coordinates)

$$\frac{\partial \underline{H}}{\partial t} = \text{curl}(\underline{v} \times \underline{H}) + \nu_e \nabla^2 \underline{H} \quad (17)$$

which is the equation of motion governing magnetic field. Equation (17) is general; it is not restricted either to incompressible fluids or to inviscid fluids.

The case when the electrical conductivity of the medium may be considered as infinite is a particular interest in cosmic electrodynamics. The magnetic viscosity is then zero, and equations (16) and (17) reduce respectively to

$$\frac{\partial \underline{A}}{\partial t} = \underline{v} \times \text{curl } \underline{A} - \text{grad } \phi \quad (18)$$

$$\frac{\partial \underline{H}}{\partial t} = \text{curl}(\underline{v} \times \underline{H}) \quad (19)$$

Equations (18) and (19), especially the latter, have been the object of considerable research in the literature. We may notice at once its full analogy with the Helmholtz equation for the vorticity. This immediately permits to apply mutatis mutandis the classical and elegant results of the theory of vorticity to the magnetic field. In particular, it follows that the lines of force move with the fluid. For further details of this analogy we refer to Goldstein's Lectures on Fluid Mechanics,<sup>(14)</sup> page 76.

### 3. The Elsasser-Carstou Theorem

Equation (19) may be put in a form which is reminiscent of the Cauchy-Stokes decomposition of an arbitrary instantaneous continuous motion of a fluid (see for instance Truesdell,<sup>(23)</sup> page 65). Equivalent to the basic equation (19) is

$$\frac{d}{dt} \left[ \frac{H_i}{\rho} \right] = \frac{H_j}{\rho} \frac{\partial v_i}{\partial x_j} \quad (20)$$

where the equation of continuity (11) has been used. Equation (20) can be rewritten

$$\frac{d}{dt} \left[ \frac{H_i}{\rho} \right] = \frac{1}{2} \frac{H_j}{\rho} \left[ \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right] + \frac{1}{2} \frac{H_j}{\rho} \left[ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right] \quad (21)$$

where, besides the rate of deformation  $e_{ij}$ , the vorticity  $\omega = \omega_{ij}$ :

$$\omega_{ij} = \frac{1}{2} \left[ \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right] \quad (22)$$

appear under its tensor components. We can write

$$\frac{d}{dt} \left[ \frac{H_i}{\rho} \right] = \frac{H_j}{\rho} \omega_{ij} + \frac{1}{2\rho} \frac{\partial G}{\partial H_i} \quad (23)$$

or in vector notation

$$\frac{d}{dt} \left[ \frac{\underline{H}}{\rho} \right] = \underline{\omega} \times \frac{\underline{H}}{\rho} + \frac{1}{2\rho} \text{Grad}_{H_1} G \quad (24)$$

where we set

$$G = e_{ij} H_i H_j \quad (25)$$

and the gradient is taken with respect to  $H_1$ .

Equation (24) shows that the rate of change of the magnetic field may be conceived as made up of two parts. The first part expresses a rotation of the field with the fluid particle; the second part shows that the terminus of  $\underline{H}$  is moving in the direction of the normal of the quadric of the system:

$$e_{ij} x_i x_j = \text{const.} \quad (26)$$

on which its terminus lies.

In this form the theorem has been stated by the writer.<sup>(4)</sup> An integral formulation closely related to this has been earlier given by Elsasser.<sup>(11,12)</sup>

#### 4. The Analogue of the Clebsch Transformation

A matter of interest in cosmic electrodynamics is the analogue of the Clebsch transformation of the hydrodynamical equations (see Lamb,<sup>(16)</sup> page 248). Putting

$$\underline{A} = \text{grad } P + \phi \text{ grad } \psi \quad (27)$$

one has

$$\underline{H} = \text{curl}(\phi \text{ grad } \psi) = \text{grad } \phi \times \text{grad } \psi \quad (28)$$

The representation (28) is identical to that given by Sweet<sup>(21)</sup> (see also Dungey,<sup>(10)</sup> page 31) with the exception of a factor  $F$ , function of  $\phi$  and  $\psi$  only, which appears on the right side of (28) in Sweet's representation. However, it can be shown that Sweet's and our representation are equivalent (see Lamb, loc. cit.).

The immediate consequences of formula (28) namely

$$\underline{H} \cdot \text{grad } \phi = \underline{H} \cdot \text{grad } \psi = 0 \quad (29)$$

show that the magnetic field is tangent to the surfaces  $\phi = \text{const.}$  and  $\psi = \text{const.}$ , which we shall call surfaces of force, and which correspond to vortex surfaces in hydrodynamics. It is evident that their intersections are the lines of force.

# 5. Hamiltonian Form for the Equation of Motion for the Magnetic Field

Let us now come back to equation (18) and substitute thereto the value of  $\underline{A}$  given by (27). We have

$$\begin{aligned} \frac{\partial}{\partial t} (\text{grad } P + \phi \text{ grad } \psi) &= \underline{v} \times (\text{grad } \phi \times \text{grad } \psi) - \text{grad } \phi \\ &= (\underline{v} \cdot \text{grad } \psi) \text{grad } \phi - (\underline{v} \cdot \text{grad } \phi) \text{grad } \psi - \text{grad } \phi \end{aligned} \quad (30)$$

which can be written

$$\frac{d\phi}{dt} \text{grad } \psi - \frac{d\psi}{dt} \text{grad } \phi = -\text{grad } \mathcal{H} \quad (31)$$

where

$$\mathcal{H} = \frac{\partial P}{\partial t} + \phi \frac{\partial \psi}{\partial t} + \phi \quad (32)$$

Scalar multiplication of terms of equation (31) by  $\text{grad } \phi \times \text{grad } \psi$  gives

$$\text{grad } \mathcal{H} \cdot (\text{grad } \phi \times \text{grad } \psi) = 0 \quad (33)$$

that is the Jacobian

$$\frac{\partial(\mathcal{H}, \phi, \psi)}{\partial(x, y, z)} = 0 \quad (34)$$

This shows that  $\mathcal{H}$  is of the form  $\mathcal{H}(\phi, \psi, t)$ . Hence

$$\text{grad } \mathcal{H} = \frac{\partial \mathcal{H}}{\partial \psi} \text{grad } \psi + \frac{\partial \mathcal{H}}{\partial \phi} \text{grad } \phi \quad (35)$$

Comparison of equations (31) and (35) gives at once the Hamiltonian system

$$\frac{d\phi}{dt} = - \frac{\partial \mathcal{H}}{\partial \psi} ; \quad \frac{d\psi}{dt} = \frac{\partial \mathcal{H}}{\partial \phi} \quad (36)$$

Equations (36) are analogous to Stuart's equations in hydrodynamics (see Lamb, loc. cit.) and were derived by this writer in a recent paper.<sup>(6)</sup>

## CHAPTER II

## MAGNETOHYDRODYNAMIC WAVES IN A CONSTANT AND UNIFORM MAGNETIC FIELD

6. The Case of an Incompressible Fluid

We begin with a discussion due to Walén<sup>(24)</sup> (see also Cowling<sup>(7,8)</sup>). Consider an infinite mass of uniform fluid, at rest, embedded in a constant and uniform magnetic field  $\underline{H}_0$ . We assume this fluid to be an inviscid, incompressible and perfectly conducting material ( $\sigma \rightarrow \infty$ ). Suppose that as a result of a perturbation, a velocity field  $\underline{v}$  is produced in a certain region, and that the magnetic field becomes  $\underline{H}_0 + \underline{h}$ . The equations giving the variations in  $\underline{v}$  and  $\underline{h}$  are

$$\rho_0 \frac{\partial \underline{v}}{\partial t} = -\text{grad } p + \rho_0 \underline{g} + \frac{\mu_e}{4\pi} \text{curl } \underline{h} \times (\underline{H}_0 + \underline{h}) \quad (1)$$

$$\frac{\partial \underline{h}}{\partial t} = \text{curl} \left[ \underline{v} \times (\underline{H}_0 + \underline{h}) \right] \quad (2)$$

where we have included the gravitational potential

$$\underline{g} = -\text{grad } \Omega \quad (3)$$

$\rho_0$  is the uniform density of our fluid, and the term  $(\underline{v} \cdot \nabla) \underline{v}$  has been omitted.

Now, since  $\underline{H}_0 = \text{constant}$  and

$$\text{div } \underline{v} = 0 \quad (4)$$

$$\text{div } \underline{h} = 0 \quad (5)$$

equations (1) and (2) simplify to

$$\rho_0 \frac{\partial \underline{v}}{\partial t} = -\text{grad} \left[ p + \frac{\mu_e \underline{H}_0 \cdot \underline{h}}{4\pi} + \rho_0 \Omega \right] + \frac{\mu_e}{4\pi} (\underline{H}_0 \cdot \nabla) \underline{h} \quad (6)$$

$$\frac{\partial \underline{h}}{\partial t} = (\underline{H}_0 \cdot \nabla) \underline{v} \quad (7)$$

by neglecting squares and products of the small quantities  $\underline{h}, \underline{v}$ . Take the divergence of equation (6); we have

$$\text{div grad} \left[ p + \frac{\mu_{eH_0} \cdot \underline{h}}{4\pi} + \rho_o \Omega \right] = 0 \quad (8)$$

In Cartesian coordinates equation (8) becomes

$$\nabla^2 \left[ p + \frac{\mu_{eH_0} \cdot \underline{h}}{4\pi} + \rho_o \Omega \right] = 0 \quad (9)$$

since

$$p + \frac{\mu_{eH_0} \cdot \underline{h}}{4\pi} + \rho_o \Omega$$

has no singularities and is bounded,

$$p + \frac{\mu_{eH_0} \cdot \underline{h}}{4\pi} + \rho_o \Omega = \text{constant} \quad (10)$$

Hence, equation (6) becomes

$$4\pi\rho_o \frac{\partial \underline{v}}{\partial t} = \mu_e (\underline{H}_o \cdot \nabla) \underline{h} \quad (11)$$

For simplicity take OZ parallel to  $\underline{H}_o$ . Then equations (11) and (7) become

$$4\pi\rho_o \frac{\partial \underline{v}}{\partial t} = \mu_e H_o \frac{\partial \underline{h}}{\partial z} \quad (12)$$

$$\frac{\partial \underline{h}}{\partial t} = H_o \frac{\partial \underline{v}}{\partial z} \quad (13)$$

Hence, by cross differentiation

$$\frac{\partial^2 \underline{v}}{\partial t^2} = A_o^2 \frac{\partial^2 \underline{v}}{\partial z^2} \quad (14)$$

$$\frac{\partial^2 \underline{h}}{\partial t^2} = A_o^2 \frac{\partial^2 \underline{h}}{\partial z^2} \quad (15)$$

where

$$A_o^2 = \frac{\mu_e H_o^2}{4\pi\rho_o} \quad (16)$$

is the Alfven's phase velocity, named so in honor of its discoverer. Thus the disturbance can be expressed as the resultant of two sets of waves traveling with velocities  $\pm A_o$  in the z-direction, i.e., along the lines of force of the undisturbed field. These waves are called magnetohydrodynamic (m.h.) waves.

After the two waves have separated we have in either of the waves

$$\frac{\partial \underline{v}}{\partial t} = \pm A_o \frac{\partial \underline{v}}{\partial z} \quad (17)$$

the sign depending on the direction of propagation of the wave considered. Comparison of equations (13) and (17) gives

$$\underline{h} = \pm \frac{H_o \underline{v}}{A_o} = \pm \sqrt{\frac{4\pi\rho_o}{\mu_e}} \underline{v} \quad (18)$$

Before going farther, we note, that in considering the propagation described by Alfven, the velocity  $\underline{v}$  and the magnetic field  $\underline{h}$  can be replaced by the vorticity  $\underline{\omega}$  and the current density  $\underline{j} = (1/4\pi)\text{curl } \underline{h}$  respectively; for one has similar equations for these quantities, namely

$$\frac{\partial^2 \underline{\omega}}{\partial t^2} = A_o^2 \frac{\partial^2 \underline{\omega}}{\partial z^2} \quad (19)$$

$$\frac{\partial^2 \underline{j}}{\partial t^2} = A_o^2 \frac{\partial^2 \underline{j}}{\partial z^2} \quad (20)$$

together with the relation

$$\underline{j} = \pm \frac{H_o}{2\pi A_o} \underline{\omega} = \pm \sqrt{\frac{\rho_o}{\pi\mu_e}} \underline{\omega} \quad (21)$$

## 7. Compressible Fluid - Vorticity and Current Density Propagation

In taking this point of view as a point of departure, we shall show that in the case of a compressible medium the components of  $\underline{\omega}$  and  $\underline{j}$  in the direction of the field



only are propagated in Alfvén's manner. Thus, surprisingly enough, the compressibility of a medium acts as a wave filter discriminating between components of vorticity and current density and passing only those directed along the (undisturbed) magnetic field. The proof goes like this. When the compressibility is taken into account, the linearized system replacing equations (6) and (7) is

$$\rho_0 \frac{\partial \underline{v}}{\partial t} = -\text{grad } \phi + \frac{\mu_e H_0}{4\pi} \frac{\partial \underline{h}}{\partial z} \quad (22)$$

$$\frac{\partial \underline{h}}{\partial t} = H_0 \frac{\partial \underline{v}}{\partial z} - H_0 \text{div } \underline{v} \quad (23)$$

where we set

$$\phi = p + \frac{\mu_e H_0 \cdot \underline{h}}{4\pi} + \rho_0 \Omega \quad (24)$$

Equations (4) and (5) are replaced by

$$\frac{\partial \rho}{\partial t} + \rho_0 \text{div } \underline{v} = 0 \quad (25)$$

$$\text{div } \underline{h} = 0 \quad (26)$$

where  $\rho$  is the perturbation in density. We shall assume that

$$p = a_0^2 \rho \quad (27)$$

where  $a_0$  is the ordinary sound speed in the absence of a magnetic field.

Taking the curl of terms of equations (22) and (23) we obtain

$$2\rho_0 \frac{\partial \underline{\omega}}{\partial t} = \mu_e H_0 \frac{\partial \underline{j}}{\partial z} \quad (28)$$

$$4\pi \frac{\partial \underline{j}}{\partial t} = 2H_0 \frac{\partial \underline{\omega}}{\partial z} + \underline{H}_0 \times \text{grad div } \underline{v} \quad (29)$$

which imply important consequences, as will be shown.

(a) Propagation of z-Components of Vorticity and Current Density

Equations (28) and (29) when projected on the  $O_z$  axis give

$$2\rho_0 \frac{\partial \omega_z}{\partial t} = \mu_e H_0 \frac{\partial j_z}{\partial z} \quad (30)$$

$$2\pi \frac{\partial j_z}{\partial t} = H_0 \frac{\partial \omega_z}{\partial z} \quad (31)$$

Hence

$$\frac{\partial^2 \omega_z}{\partial t^2} = A_0^2 \frac{\partial^2 \omega_z}{\partial z^2} \quad (32)$$

$$\frac{\partial^2 j_z}{\partial t^2} = A_0^2 \frac{\partial^2 j_z}{\partial z^2} \quad (33)$$

and

$$j_z = \pm \frac{H_0}{2\pi A_0} \omega_z \quad (34)$$

Thus, the components of  $\omega$  and  $j$  along the lines of force (longitudinal components) are propagated in the opposite directions of the undisturbed field with velocities  $\pm A_0$ .

The coupling relationship (37) between longitudinal components shows that (i) it does not depend on the magnitude of the magnetic field present; (ii) the vanishing of either component involves the vanishing of the other; this occurs when either quantity is zero initially.

(b) Equations for the Transverse Components

It may be noted that although  $\omega_z$  and  $j_z$  are propagated one-dimensionally, along the magnetic lines of force, no other component of vorticity and current density is; the x- and y-components of vorticity and current density satisfy

$$\frac{\partial^2 \omega_x}{\partial t^2} - A_o^2 \frac{\partial^2 \omega_x}{\partial z^2} = \frac{A_o^2}{2\rho_o} \frac{\partial^3 \rho}{\partial t \partial y \partial z} \quad (35)$$

$$\frac{\partial^2 \omega_y}{\partial t^2} - A_o^2 \frac{\partial^2 \omega_y}{\partial z^2} = - \frac{A_o^2}{2\rho_o} \frac{\partial^3 \rho}{\partial t \partial x \partial z} \quad (36)$$

and

$$\frac{\partial^2 j_x}{\partial t^2} - A_o^2 \frac{\partial^2 j_x}{\partial z^2} = \frac{H_o}{4\pi\rho_o} \frac{\partial^3 \rho}{\partial t^2 \partial y} \quad (37)$$

$$\frac{\partial^2 j_y}{\partial t^2} - A_o^2 \frac{\partial^2 j_y}{\partial z^2} = - \frac{H_o}{4\pi\rho_o} \frac{\partial^3 \rho}{\partial t^2 \partial x} \quad (38)$$

showing that only for incompressible flow do they satisfy the same equations as  $\omega_z$  and  $j_z$ ; in a compressible flow their oscillations are coupled to those of density.

#### 8. Wave-Motion Equations for the Density and Transverse Components

Differentiation with respect to  $t$  of equation (25) gives

$$\frac{\partial^2 \rho}{\partial t^2} = a_o^2 \nabla^2 \rho + \frac{A_o^2 \rho_o}{H_o} \nabla^2 h_z \quad (39)$$

where the equations (22) and (27) have been used, and the gravitational potential has been omitted. Now

$$\nabla^2 h_z = 4\pi \left[ \frac{\partial j_x}{\partial y} - \frac{\partial j_y}{\partial x} \right] \quad (40)$$

Hence, the density satisfies

$$\frac{\partial^2 \rho}{\partial t^2} = a_o^2 \nabla^2 \rho + \frac{4\pi A_o^2 \rho_o}{H_o} \left[ \frac{\partial j_x}{\partial y} - \frac{\partial j_y}{\partial x} \right] \quad (41)$$

To obtain an equation for  $\rho$  alone, we eliminate  $j_x$  and  $j_y$  between equations (37), (38), and (41). The result is

$$\frac{\partial^2}{\partial t^2} \left[ \frac{\partial^2 \rho}{\partial t^2} - a_o^2 \nabla^2 \rho \right] - A_o^2 \frac{\partial^2}{\partial z^2} \left[ \frac{\partial^2 \rho}{\partial t^2} - a_o^2 \nabla^2 \rho \right] = A_o^2 \frac{\partial^2}{\partial t^2} \left[ \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right] \quad (42)$$

where a two-dimensional Laplacian appears. This equation shows plainly the radical departure of the "new sound-wave equation" from the ordinary sound equation in the absence of a magnetic field.

We may ask now if it were possible to satisfy both equations (42) and

$$\frac{\partial^2 \rho}{\partial t^2} - a_o^2 \nabla^2 \rho = 0 \quad (43)$$

Then, by virtue of the former, we must have

$$\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} = 0 \quad (44)$$

and hence equation (43) reduces to

$$\frac{\partial^2 \rho}{\partial t^2} - a_o^2 \frac{\partial^2 \rho}{\partial z^2} = 0 \quad (45)$$

which admits a solution of the form

$$\rho \cong e^{i(\omega t - \gamma z)} \quad (46)$$

provided that

$$\gamma^2 = \frac{\omega^2}{a_o^2} \quad (47)$$

We can now easily verify that (46) is effectively a particular solution of equation (42) under condition (47). Thus sound waves appear possible in a conducting fluid penetrated by a uniform magnetic field, with this great difference that they do not spread out three-dimensionally as in ordinary acoustics; instead, they propagate (without attenuation) one-dimensionally, along the magnetic lines of force. It is also interesting to note that in contrast to m.h. waves, this propagation does not depend on the magnitude of the magnetic field present.

Equation (42) can be rewritten as follows

$$\frac{\partial^2}{\partial t^2} \left[ \frac{\partial^2 \rho}{\partial t^2} - (a_o^2 + A_o^2) \nabla^2 \rho \right] + a_o^2 A_o^2 \frac{\partial^2}{\partial z^2} \nabla^2 \rho = 0 \quad (48)$$

Differentiation with respect to  $t$  of terms of equation (48) yields an equation given by Lighthill<sup>(17)</sup> for the expansion  $\Delta = \text{div } \underline{y}$ .

Let us come back to equations (35) - (38) and improve our results. Elimination of  $\rho$  between equations (35) and (48) and then between (36) and (48) gives

$$\left\{ \frac{\partial^2}{\partial t^2} \left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial t^2} - (a_o^2 + A_o^2) \nabla^2 \right) + a_o^2 A_o^2 \frac{\partial^2}{\partial z^2} \nabla^2 \right] - A_o^2 \frac{\partial^2}{\partial z^2} \left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial t^2} - (a_o^2 - A_o^2) \nabla^2 \right) + a_o^2 A_o^2 \frac{\partial^2}{\partial z^2} \nabla^2 \right] \right\}_{\omega_y}^{\omega_x} = 0 \quad (49)$$

Similarly, elimination of  $\rho$  between equations (37), (38) and (48) gives

$$\left\{ \frac{\partial^2}{\partial t^2} \left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial t^2} - (a_o^2 + A_o^2) \nabla^2 \right) + a_o^2 A_o^2 \frac{\partial^2}{\partial z^2} \nabla^2 \right] - A_o^2 \frac{\partial^2}{\partial z^2} \left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial t^2} - (a_o^2 + A_o^2) \nabla^2 \right) + a_o^2 A_o^2 \frac{\partial^2}{\partial z^2} \nabla^2 \right] \right\}_{j_y}^{j_x} = 0 \quad (50)$$

Equations (49) and (50) show that the quantities

$$\left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial t^2} - (a_o^2 + A_o^2) \nabla^2 \right) + a_o^2 A_o^2 \frac{\partial^2}{\partial z^2} \nabla^2 \right]_{j_x, j_y}^{\omega_x, \omega_y} \quad (51)$$

are propagated along magnetic lines of force at Alfvén velocity  $A_o$ . These quantities are identical zero if they were zero initially; under this condition, equations (49) and (50) reduce to fourth-order equations of the same type as equation (48).

## CHAPTER III

## THE GEOMETRY OF DIPOLE MAGNETIC LINES OF FORCE

9. Preliminaries

The equations of motion in the case of a constant magnetic dipole are complicated and require some detailed discussion of the geometry of the magnetic lines of force. Störmer<sup>(22)</sup> was the first to use in his equations the arc  $s$  of a line of force, instead of the time (see his book, page 215). He also calculated the radius of curvature of the trajectories in the equatorial plane of the dipole (loc. cit., page 221). Also, other authors used at least the element of length of magnetic lines of force in various of their calculations. However, nowhere do we find a detailed discussion of the geometry of these lines. In this Chapter, we propose to fulfill this need by a systematic account of the geometry of magnetic lines of force for the case under consideration.

10. Equations of Lines of Force and Linear Element

As well known, a dipole magnetic field has components

$$H_x = -M \frac{3xz}{r^5}, \quad H_y = -M \frac{3yz}{r^5}, \quad H_z = -M \frac{r^2 - 3z^2}{r^5} \quad (1)$$

where  $M$  is the magnetic moment of the dipole,  $r^2 = x^2 + y^2 + z^2$ , and the sign is chosen such that  $H_z$  is positive in the  $x, y$  plane, which is the equatorial plane of the dipole; this requires that the dipole has its negative pole directed upward.

The differential equations of lines of force are

$$\frac{dx}{-3xz} = \frac{dy}{-3yz} = \frac{dz}{x^2 + y^2 - 2z^2} \quad (2)$$

The first two equations give at once

$$y = Cx \quad (3)$$

where  $C$  is a constant. Equation (3) represents a family of planes passing through  $OZ$  axis.

Then, by virtue of (3), the last two equations give

$$\frac{dz}{dx} = \frac{2z^2 - A^2 x^2}{3xz} \quad (4)$$

$$A^2 = 1 + C^2 \quad (5)$$

This is a homogeneous equation of the first order. On substitution of

$$z = xu \quad (6)$$

the equation becomes

$$\frac{dx}{x} = - \frac{3udu}{u^2 + A^2} \quad (7)$$

in which the variables are separated; the solution is

$$x^{-\frac{2}{3}} = B(u^2 + A^2) \quad (8)$$

B being a constant of integration. Hence

$$x^{\frac{4}{3}} = B(x^2 + y^2 + z^2) \quad (9)$$

that is

$$x^{\frac{2}{3}} = ar \quad (10)$$

where  $a = \sqrt{B}$ .

We now introduce the polar coordinates

$$\left. \begin{aligned} x &= r \cos \phi \cos \lambda \\ y &= r \sin \phi \cos \lambda \\ z &= r \sin \lambda \end{aligned} \right\} \quad (11)$$

where  $\lambda$  designates the magnetic latitude. In these coordinates, equation (10) becomes

$$\cos^2 \phi \cos^2 \lambda = a^3 r \quad (12)$$

To eliminate  $\phi$ , we write

$$\frac{y}{x} = C = \tan \phi \quad (13)$$

Hence

$$\cos^2 \phi = \frac{1}{C^2 + 1} \quad (14)$$

and equation (12) becomes

$$r = r_0 \cos^2 \lambda \quad (15)$$

where we put  $r_0 = 1/[a^3(C^2 + 1)]$ . Equation (15) is the equation of lines of force in each meridian plane; it is obvious that  $r_0$  is the value of  $r$  for  $\lambda = 0$  (in equatorial plane).

In Cartesian coordinates, we have the following parametric equations of the lines of force

$$\left. \begin{aligned} x &= \frac{r \cos \lambda}{\sqrt{C^2 + 1}} \\ y &= \frac{Cr \cos \lambda}{\sqrt{C^2 + 1}} \\ z &= r \sin \lambda \end{aligned} \right\} \quad (16)$$

To calculate the linear element of these lines we may use either equation (15) and then

$$ds^2 = dr^2 + r^2 d\lambda^2 \quad (17)$$

or the parametric equations (16) and then we have

$$ds^2 = dx^2 + dy^2 + dz^2 \quad (18)$$

The result is



$$ds = r d\lambda (1 + 4 \tan^2 \lambda)^{\frac{1}{2}} \quad (19)$$

# 11. The Frenet Formulas for a Line of Force

The tangent to a line of force is defined by

$$\left. \begin{aligned} \alpha = \frac{dx}{ds} &= - \frac{3}{\sqrt{c^2 + 1}} \frac{\sin \lambda}{\sqrt{1 + 4 \tan^2 \lambda}} \\ \beta = \frac{dy}{ds} &= - \frac{3c}{\sqrt{c^2 + 1}} \frac{\sin \lambda}{\sqrt{1 + 4 \tan^2 \lambda}} \\ \gamma = \frac{dz}{ds} &= \frac{(1 - 2 \tan^2 \lambda) \cos \lambda}{\sqrt{1 + 4 \tan^2 \lambda}} \end{aligned} \right\} \quad (20)$$

We next calculate the quantities  $d\alpha/ds$ ,  $d\beta/ds$ ,  $d\gamma/ds$ . One has for instance,

$$\frac{d\alpha}{ds} = \frac{d\alpha}{d\lambda} \frac{d\lambda}{ds}, \text{ etc.}$$

After some calculation, we obtain

$$\left. \begin{aligned} \frac{d\alpha}{ds} &= - \frac{3 \cos \lambda}{\sqrt{c^2 + 1}} \frac{1 - 4 \tan^2 \lambda}{r(1 + 4 \tan^2 \lambda)^2} \\ \frac{d\beta}{ds} &= \frac{-3c \cos \lambda}{\sqrt{c^2 + 1}} \frac{1 - 4 \tan^2 \lambda}{r(1 + 4 \tan^2 \lambda)^2} \\ \frac{d\gamma}{ds} &= \frac{-9 \sin \lambda (1 + 2 \tan^2 \lambda)}{r(1 + 4 \tan^2 \lambda)^2} \end{aligned} \right\} \quad (21)$$

The radius of curvature  $\rho$  is given by

$$\begin{aligned} \frac{1}{\rho^2} &= \left[ \frac{d\alpha}{ds} \right]^2 + \left[ \frac{d\beta}{ds} \right]^2 + \left[ \frac{d\gamma}{ds} \right]^2 \\ &= \frac{9(1 + 2 \tan^2 \lambda)^2}{r^2(1 + 4 \tan^2 \lambda)^3} \end{aligned} \quad (22)$$

In the equatorial plane

$$\rho = \frac{r}{3} = \frac{r_0}{3} \quad (23)$$

As a verification of result (22), we may use the formula

$$\rho = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{r^2 + 2r'^2 - rr''} \quad (24)$$

One has

$$\left. \begin{aligned} r &= r_0 \cos^2 \lambda \\ r' &= -2r_0 \cos \lambda \sin \lambda \\ r'' &= -2r_0 (\cos^2 \lambda - \sin^2 \lambda) \end{aligned} \right\} \quad (25)$$

Hence

$$\begin{aligned} \rho &= \frac{r_0^3 (\cos^4 \lambda + 4 \cos^2 \lambda \sin^2 \lambda)^{\frac{3}{2}}}{\cos^2 \lambda [\cos^2 \lambda + 8 \sin^2 \lambda + 2(\cos^2 \lambda - \sin^2 \lambda)]} \\ &= \frac{r_0 \cos^2 \lambda (1 + 4 \tan^2 \lambda)^{\frac{3}{2}}}{3(1 + 2 \tan^2 \lambda)} \end{aligned} \quad (26)$$

which checks our earlier result (22).

The principal normal, which in our case reduces to the normal of lines of force as these are plane curves, is determined by

$$\begin{aligned} \alpha_1 &= \rho \frac{d\alpha}{ds} = - \frac{\cos \lambda}{\sqrt{c^2 + 1}} \frac{1 - 2 \tan^2 \lambda}{\sqrt{1 + 4 \tan^2 \lambda}} \\ \beta_1 &= \rho \frac{d\beta}{ds} = - \frac{c \cos \lambda}{\sqrt{c^2 + 1}} \frac{1 - 2 \tan^2 \lambda}{\sqrt{1 + 4 \tan^2 \lambda}} \\ \gamma_1 &= \rho \frac{d\gamma}{ds} = - \frac{3 \sin \lambda}{\sqrt{1 + 4 \tan^2 \lambda}} \end{aligned} \quad (27)$$

The binormal of lines of force is, of course, the unit normal to the planes  $y = Cx$ ; therefore, its direction causes are

$$\alpha_2 = \frac{C}{\sqrt{C^2 + 1}}, \quad \beta_2 = -\frac{1}{\sqrt{C^2 + 1}}, \quad \gamma_2 = 0 \quad (28)$$

Therefore, direct application of the Frenet formulas has to give the above results. This will verify our previous results. One has

$$\begin{aligned} \alpha_2 &= \beta\gamma_1 - \gamma\beta_1 = \frac{C}{\sqrt{C^2 + 1}} \frac{1}{1 + 4 \tan^2 \lambda} \left[ 9 \sin^2 \lambda + \cos^2 \lambda (1 - 2 \tan^2 \lambda)^2 \right] \\ &= \frac{C}{\sqrt{C^2 + 1}} \frac{\cos^2 \lambda}{1 + 4 \tan^2 \lambda} (1 + 5 \tan^2 \lambda + 4 \tan^4 \lambda) \\ &= \frac{C}{\sqrt{C^2 + 1}} \frac{\cos^2 \lambda}{1 + 4 \tan^2 \lambda} (1 + 4 \tan^2 \lambda)(1 + \tan^2 \lambda) \\ &= \frac{C}{\sqrt{C^2 + 1}} \end{aligned} \quad (29)$$

Also

$$\beta_2 = \gamma\alpha_1 - \alpha\gamma_1 = -\frac{1}{\sqrt{C^2 + 1}} \quad (30)$$

$$\gamma_2 = \alpha\beta_1 - \beta\alpha_1 = 0 \quad (31)$$

which values agree with those given by (28).

## CHAPTER IV

## MAGNETOHYDRODYNAMIC WAVES IN A CONSTANT DIPOLE MAGNETIC FIELD

12. Preliminaries

Dungey<sup>(9)</sup> in his remarkable report of 1954 has discussed in some detail the electrodynamic behavior of the Outer Atmosphere in the presence of a constant dipole magnetic field. We shall here approach the problem from a different point of view, however, concentrating our attention, as we did in Chapter II, on the vorticity field and the current density.

Consider an infinite mass of an electrically conducting fluid at rest embedded in a constant dipole magnetic field  $\underline{H}$ . To simplify the discussion, take the conductivity as infinite and assume the fluid to be a homogeneous incompressible material. Assume that as a result of a perturbation, a velocity  $\underline{v}$  is produced in a certain region and that the magnetic field becomes  $\underline{H} + \underline{h}$ . The amplitude is assumed to be small enough for non-linear terms to be neglected. We propose to investigate the magnetohydrodynamic behavior of the fluid in terms of generalized Alfvén waves.

13. Fundamental Equations

The relevant equations for the problem are

$$\rho_0 \frac{\partial \underline{v}}{\partial t} = -\text{grad } p + \mu_e \underline{j} \times \underline{H} \quad (1)$$

$$\frac{\partial \underline{h}}{\partial t} = (\underline{H} \cdot \nabla) \underline{v} - (\underline{v} \cdot \nabla) \underline{H} \quad (2)$$

$$\text{div } \underline{v} = 0 \quad (3)$$

$$\text{div } \underline{h} = 0 \quad (4)$$

the condition  $\partial \underline{H} / \partial t = 0$  (constant dipole) has been used in equation (2). Since  $\text{curl } \underline{H} = 0$ , equation (1) can be rewritten as follows

$$\rho_0 \frac{\partial \underline{v}}{\partial t} = -\text{grad} \left[ p + \frac{\mu_e \underline{H} \cdot \underline{h}}{4\pi} \right] + \frac{\mu_e}{4\pi} \left[ (\underline{H} \cdot \nabla) \underline{h} + (\underline{h} \cdot \nabla) \underline{H} \right] \quad (5)$$

Taking the curl of terms of equations (5) and (2) we obtain

$$2\rho_0 \frac{\partial \omega}{\partial t} = \frac{\mu_e}{4\pi} \text{curl} \left[ (\underline{H} \cdot \nabla) \underline{h} + (\underline{h} \cdot \nabla) \underline{H} \right] \quad (6)$$

$$4\pi \frac{\partial \underline{j}}{\partial t} = \text{curl} \left[ (\underline{H} \cdot \nabla) \underline{v} - (\underline{v} \cdot \nabla) \underline{H} \right] \quad (7)$$

We have

$$\begin{aligned} \text{curl} \left[ (\underline{H} \cdot \nabla) \underline{h} + (\underline{h} \cdot \nabla) \underline{H} \right] &= 4\pi (\underline{H} \cdot \nabla) \underline{j} + \text{grad } h_x \times \frac{\partial \underline{H}}{\partial x} + \text{grad } h_y \times \frac{\partial \underline{H}}{\partial y} \\ &\quad + \text{grad } h_z \times \frac{\partial \underline{H}}{\partial z} + \text{grad } H_x \times \frac{\partial \underline{h}}{\partial x} + \text{grad } H_y \times \frac{\partial \underline{h}}{\partial y} \\ &\quad + \text{grad } H_z \times \frac{\partial \underline{h}}{\partial z} \end{aligned} \quad (8)$$

After some calculation, we obtain

$$\text{curl} \left[ (\underline{H} \cdot \nabla) \underline{h} + (\underline{h} \cdot \nabla) \underline{H} \right] = 4\pi \left[ (\underline{H} \cdot \nabla) \underline{j} - (\underline{j} \cdot \nabla) \underline{H} \right] \quad (9)$$

where the condition  $\text{curl } \underline{H} = 0$  has been used.

Hence equation (6) can be written

$$2\rho_0 \frac{\partial \omega}{\partial t} = \mu_e \left[ (\underline{H} \cdot \nabla) \underline{j} - (\underline{j} \cdot \nabla) \underline{H} \right] \quad (10)$$

On the other hand,

$$\begin{aligned} \text{curl} \left[ (\underline{H} \cdot \nabla) \underline{v} - (\underline{v} \cdot \nabla) \underline{H} \right] &= 2(\underline{H} \cdot \nabla) \underline{\omega} + \text{grad } H_x \times \frac{\partial \underline{v}}{\partial x} + \text{grad } H_y \times \frac{\partial \underline{v}}{\partial y} \\ &\quad + \text{grad } H_z \times \frac{\partial \underline{v}}{\partial z} - \left[ \text{grad } v_x \times \frac{\partial \underline{H}}{\partial x} \right. \\ &\quad \left. + \text{grad } v_y \times \frac{\partial \underline{H}}{\partial y} + \text{grad } v_z \times \frac{\partial \underline{H}}{\partial z} \right] \end{aligned} \quad (11)$$

In this case, there is, however, no simple way to write vectorially equation (7) in a compact form; we have for components the following equations

$$\begin{aligned}
 2\pi \frac{\partial j_x}{\partial t} &= \left[ (\underline{H} \cdot \nabla) \underline{\omega} \right]_x + \frac{\partial H_x}{\partial y} e_{31} + \frac{\partial H_y}{\partial y} e_{32} + \frac{\partial H_z}{\partial y} e_{33} - \left[ \frac{\partial H_x}{\partial z} e_{21} + \frac{\partial H_y}{\partial z} e_{22} + \frac{\partial H_z}{\partial z} e_{23} \right] \\
 2\pi \frac{\partial j_y}{\partial t} &= \left[ (\underline{H} \cdot \nabla) \underline{\omega} \right]_y + \frac{\partial H_x}{\partial z} e_{11} + \frac{\partial H_y}{\partial z} e_{12} + \frac{\partial H_z}{\partial z} e_{13} - \left[ \frac{\partial H_x}{\partial x} e_{31} + \frac{\partial H_y}{\partial x} e_{32} + \frac{\partial H_z}{\partial x} e_{33} \right] \\
 2\pi \frac{\partial j_z}{\partial t} &= \left[ (\underline{H} \cdot \nabla) \underline{\omega} \right]_z + \frac{\partial H_x}{\partial x} e_{12} + \frac{\partial H_y}{\partial x} e_{22} + \frac{\partial H_z}{\partial x} e_{23} - \left[ \frac{\partial H_x}{\partial y} e_{11} + \frac{\partial H_y}{\partial y} e_{12} + \frac{\partial H_z}{\partial y} e_{13} \right]
 \end{aligned}
 \tag{12}$$

where  $e_{ij}$  is the rate of deformation and where again this condition  $\text{curl } \underline{H} = 0$  has been used.

#### 14. Propagation at Large Distances

These equations are rather complicated. We may simplify them by observing that the derivatives of the components of the dipole magnetic field are the order of  $r^{-4}$  while these components themselves are of the order  $r^{-3}$ . Therefore for  $r$  sufficiently large, we may neglect the term  $(\underline{j} \cdot \nabla) \underline{H}$  in equation (10) and similarly we may neglect all terms such as  $(\partial H_x / \partial y) e_{31}$ , etc., in equation (12). Hence, for  $r$  sufficiently large, equations (10) and (12) reduce to

$$2\rho_o \frac{\partial \underline{\omega}}{\partial t} = \mu_e \left[ H_x \frac{\partial \underline{j}}{\partial x} + H_y \frac{\partial \underline{j}}{\partial y} + H_z \frac{\partial \underline{j}}{\partial z} \right] \tag{13}$$

and

$$2\pi \frac{\partial \underline{j}}{\partial t} = H_x \frac{\partial \underline{\omega}}{\partial x} + H_y \frac{\partial \underline{\omega}}{\partial y} + H_z \frac{\partial \underline{\omega}}{\partial z} \tag{14}$$

In the second place, we have a long a line of force

$$H_x = H \frac{dx}{ds}, \quad H_y = H \frac{dy}{ds}, \quad H_z = H \frac{dz}{ds} \tag{15}$$

where  $H$  is the **magnitude** of the dipole magnetic field and  $ds$  the element of length of line of force. Hence along a line of force, equations (13) and (14) can be written

$$2\rho_o \frac{\partial \omega}{\partial t} = \mu_e H \left[ \frac{\partial j}{\partial x} \frac{dx}{ds} + \frac{\partial j}{\partial y} \frac{dy}{ds} + \frac{\partial j}{\partial z} \frac{dz}{ds} \right] \quad (16)$$

and

$$2\pi \frac{\partial j}{\partial t} = H \left[ \frac{\partial \omega}{\partial x} \frac{dx}{ds} + \frac{\partial \omega}{\partial y} \frac{dy}{ds} + \frac{\partial \omega}{\partial z} \frac{dz}{ds} \right] \quad (17)$$

that is

$$2\rho_o \frac{\partial \omega}{\partial t} = \mu_e H \frac{\partial j}{\partial s} \quad (18)$$

$$2\pi \frac{\partial j}{\partial t} = H \frac{\partial \omega}{\partial s} \quad (19)$$

Observing that  $\partial H / \partial t = 0$  (constant dipole), we obtain by cross-differentiation

$$\frac{\partial^2 \omega}{\partial t^2} = A^2 \frac{\partial^2 \omega}{\partial s^2} + \frac{1}{2} \frac{dA^2}{ds} \frac{\partial \omega}{\partial s} \quad (20)$$

$$\frac{\partial^2 j}{\partial t^2} = A^2 \frac{\partial^2 j}{\partial s^2} + \frac{1}{2} \frac{dA^2}{ds} \frac{\partial j}{\partial s} \quad (21)$$

where

$$\begin{aligned} A^2 &= \frac{\mu_e H^2}{4\pi\rho_o} = \frac{\mu_e M^2}{4\pi\rho_o} \frac{\cos^2 \lambda (1 + 4 \tan^2 \lambda)}{r^6} \\ &= \gamma^2 \frac{1 + 4 \tan^2 \lambda}{r_o^6 \cos^{10} \lambda} \end{aligned} \quad (22)$$

where we put

$$\gamma^2 = \frac{\mu_e M^2}{4\pi\rho_o}$$

We achieve the reduction of these equations by taking instead of  $s$  the magnetic latitude  $\lambda$  as independent variable. We have

$$\frac{\partial \omega}{\partial s} = \frac{\partial \omega}{\partial \lambda} \frac{d\lambda}{ds} \quad (23)$$

$$\frac{\partial^2 \omega}{\partial s^2} = \frac{\partial^2 \omega}{\partial \lambda^2} \left[ \frac{d\lambda}{ds} \right]^2 + \frac{\partial \omega}{\partial \lambda} \frac{d^2 \lambda}{ds^2} \quad (24)$$

and

$$\frac{d^2 \lambda}{ds^2} = \left[ \frac{d}{d\lambda} \left[ \frac{d\lambda}{ds} \right] \right] \frac{d\lambda}{ds} \quad (25)$$

Now (see Chapter III, equation (9))

$$\frac{d\lambda}{ds} = \frac{1}{r_0 \cos^2 \lambda (1 + 4 \tan^2 \lambda)^{\frac{1}{2}}} \quad (26)$$

therefore, we have

$$\frac{d^2 \lambda}{ds^2} = \frac{2 \tan \lambda (2 \tan^2 \lambda - 1)}{r_0^2 \cos^4 \lambda (1 + 4 \tan^2 \lambda)^2} \quad (27)$$

On the other hand

$$\frac{dA^2}{ds} = \frac{dA^2}{d\lambda} \frac{d\lambda}{ds} \quad (28)$$

and

$$\begin{aligned} \frac{dA^2}{ds} \frac{\partial \omega}{\partial s} &= \frac{dA^2}{d\lambda} \left[ \frac{d\lambda}{ds} \right]^2 \frac{\partial \omega}{\partial \lambda} \\ &= \frac{6\gamma^2 (3 + 8 \tan^2 \lambda) \tan \lambda}{r_0^8 \cos^{14} \lambda (1 + 4 \tan^2 \lambda)} \frac{\partial \omega}{\partial \lambda} \end{aligned} \quad (29)$$

Substitution of these values in equation (20) gives



$$\begin{aligned} \frac{\partial^2 \underline{\omega}}{\partial t^2} = & \gamma^2 \frac{1 + 4 \tan^2 \lambda}{r_0^6 \cos^{10} \lambda} \left[ \frac{\partial^2 \underline{\omega}}{\partial \lambda^2} \frac{1}{r_0^2 \cos^4 \lambda (1 + 4 \tan^2 \lambda)} + \frac{\partial \underline{\omega}}{\partial \lambda} \frac{2 \tan \lambda (2 \tan^2 \lambda - 1)}{r_0^2 \cos^4 \lambda (1 + 4 \tan^2 \lambda)^2} \right] \\ & + \frac{\partial \underline{\omega}}{\partial \lambda} \gamma^2 \frac{3 \tan \lambda (3 + 8 \tan^2 \lambda)}{r_0^8 \cos^{14} \lambda (1 + 4 \tan^2 \lambda)} \end{aligned} \quad (30)$$

that is

$$\frac{\partial^2 \underline{\omega}}{\partial t^2} = \frac{\gamma^2}{r_0^8 \cos^{14} \lambda} \left[ \frac{\partial^2 \underline{\omega}}{\partial \lambda^2} + 7 \tan \lambda \cdot \frac{\partial \underline{\omega}}{\partial \lambda} \right] \quad (31)$$

and a similar equation for  $\underline{j}$ . Supposing that

$$\underline{\omega}(\lambda, t) = A e^{i \alpha t} \underline{\omega}_1(\lambda) \quad (32)$$

we get

$$\frac{d^2 \underline{\omega}}{d\lambda^2} + 7 \tan \lambda \frac{d\underline{\omega}_1}{d\lambda} + \beta^2 r_0^8 \cos^{14} \lambda \cdot \underline{\omega}_1 = 0 \quad (33)$$

where  $\beta^2 = \alpha^2 / \gamma^2$ .

#### 15. Integral Equations for Vorticity and Current Density

Equation (33) can be transformed in an integral equation similar to that given by Dungey (see Reference 9, page 33). In order to do this we use the identity

$$\frac{d}{d\lambda} \left[ \sec \lambda \frac{d}{d\lambda} (\underline{\omega}_1 \sec^3 \lambda) \right] = \sec^4 \lambda \left[ \frac{d^2 \underline{\omega}_1}{d\lambda^2} + 7 \tan \lambda \frac{d\underline{\omega}_1}{d\lambda} + 3(1 + 5 \tan^2 \lambda) \underline{\omega}_1 \right] \quad (34)$$

Hence equation (33) can be written

$$\frac{d}{d\lambda} \left[ \sec \lambda \frac{d}{d\lambda} (\underline{\omega}_1 \sec^3 \lambda) \right] = \left[ 3(1 + 5 \tan^2 \lambda) \sec \lambda - \beta^2 r_0^8 \cos^{13} \lambda \right] \underline{\omega}_1 \sec^3 \lambda \quad (35)$$

or, by putting  $\underline{\Omega} = \underline{\omega}_1 \sec^3 \lambda$ ,

$$\frac{d}{d\lambda} \left[ \sec \lambda \frac{d\Omega}{d\lambda} \right] = \left[ 3(1 + 5 \tan^2 \lambda) \sec \lambda - \beta^2 r_o^8 \cos^{13} \lambda \right] \Omega \quad (36)$$

Assuming that for  $\lambda = 0$ ,  $d\Omega/d\lambda = 0$ , i.e.,  $d\Omega_1/d\lambda = 0$ , we obtain the following integral equation

$$\Omega(\lambda) = \Omega(0) + \int_0^\lambda \cos \lambda' d\lambda' \int_0^{\lambda'} \left[ 3(1 + 5 \tan^2 \lambda) \sec \lambda'' - \beta^2 r_o^8 \cos^{13} \lambda'' \right] \Omega(\lambda'') d\lambda'' \quad (37)$$

which, curiously enough, has the same form as the equation given by Dungey (loc. cit., page 33) but is of vectorial character of and includes an additional term  $3(1 + 5 \tan^2 \lambda) \sec \lambda$ ; the variables and assumptions used to arrive at this result differ ratically from those used by Dungey.

The quantity  $\underline{J}(\lambda) = \underline{j}_1(\lambda) \sec^3 \lambda$  verifies, of course, the same equation (37). Equation (37) may be integrated by successive approximations.

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